G_2 Perfect-Fluid Cosmologies with a proper conformal Killing vector

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February 7, 2008

Abstract

We study the Einstein field equations for spacetimes admitting a maximal two-dimensional abelian group of isometries acting orthogonally transitively on spacelike surfaces and, in addition, with at least one conformal Killing vector. The three-dimensional conformal group is restricted to the case when the two-dimensional abelian isometry subalgebra is an ideal and it is also assumed to act on non-null hypersurfaces (both, spacelike and timelike cases are studied). We consider both, diagonal and non-diagonal metrics and find all the perfect-fluid solutions under these assumptions (except those already known). We find four families of solutions, each one containing arbitrary parameters for which no differential equations remain to be integrated. We write the line-elements in a simplified form and perform a detailed study for each of these solutions, giving the kinematical quantities of the fluid velocity vector, the energy-density and pressure, values of the parameters for which the energy conditions are fulfilled everywhere, the Petrov type, the singularities in the spacetimes and the Friedmann-Lemaître-Robertson-Walker metrics contained in each family.

PACS numbers: 04.20.Jb 98.80.-k

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1 Introduction

One of the most successful ways of finding exact solutions of Einstein field equations has been the assumption of a certain degree of symmetry in the spacetime manifold [14]. Although a few solutions are known without any symmetry at all, a systematic study of exact solutions with a given isometry group has only been performed when the dimension r of the isometry group G_r is $r \geq 2$. In the particular case of G_2 spacetimes, it is usually assumed that the isometry group is abelian and acts orthogonally transitively on 2-surfaces (that is to say, that the two planes orthogonal to the group orbits at each point are themselves surface-forming). In the cosmological context, the two-dimensional orbits of the isometry group are taken spacelike (S_2) and the matter content is usually assumed to be a perfect fluid. The Einstein field equations in this situation are still very complicated and further assumptions are needed to handle the problem. They may be very different in nature, for instance: separability of the metric coefficients, degenerate Petrov types for the Weyl tensor, particular equations of state, kinematical properties of the fluid velocity vector, Kerr-Schild ansatz, etc. Among these additional simplifications, we will assume the existence of conformal symmetries in the spacetime (for a definition of conformal symmetry see e.g. [14] and for a detailed study of some general properties of conformal Killing vectors see [11]).

A conformal motion in the manifold is interesting for several reasons. First of all, it is a well-posed geometrical assumption which, therefore, does not depend on the coordinate system we are using. The Friedmann-Lemaître-Robertson-Walker (FLRW) spacetimes (which are at present our best candidates to describe the real universe, despite them being highly symmetric) possess a fifteen-dimensional conformal group. Consequently, the inhomogeneous and anisotropic models containing a conformal Killing vector (CKV) are likely to contain FLRW limits and, therefore, are suitable as generalizations of FLRW cosmologies (in particular, they can be used to test several perturbative schemes used to study inhomogeneities in the universe). Another reason for studying inhomogeneous cosmologies admitting a CKV is the Ehlers-Geren-Sachs result [10] which implies that observers measuring an isotropic microwave background radiation can exist only if the spacetime possesses a timelike CKV and the observer moves collinearly with it.

Perfect-fluid spacetimes admitting a CKV have received considerable attention in the last few years. The case in which the CKV is inheriting (meaning that the conformal motion maps the fluid world-lines into themselves) has been studied in [5]. In particular, the spherically symmetric ([6] and references therein) and the plane symmetric spacetimes [8] admitting an inheriting CKV have been fully investigated. In [7] it was shown that perfect-fluid spacetimes satisfying an equation of state $p = p(\rho)$ and admitting a CKV parallel to the fluid velocity must be FLRW models. A first attempt to perform a systematic classification of spacetimes using its conformal isometry group has been performed in [4] where some new exact solutions have also been found. All

this work showed that the perfect-fluid models admitting a CKV were rare but not impossible. In vacuum metrics, the most general spacetime admitting a CKV [9] is either Minkowski or some type N solutions (representing certain plane wave solutions).

The perfect-fluid case when the spacetime admits an abelian two-dimensional isometry group acting orthogonally transitively and one linearly independent proper conformal motion (the homothetic case was studied in [2]) has been considered only very recently [19], [3]. In these articles, the authors classify the three-dimensional conformal group according to the possible inequivalent Lie algebras and find canonical line-elements for each case. Then, they concentrate on the diagonal metrics (which already excludes a number of Lie algebras, see below) and study systematically the possible solutions of Einstein's field equations for some selected Lie algebras (in our notation below, expression (13), the Lie algebras considered in [3] correspond to Lie Algebra A with $b = 0$).

In this paper we will complete this work by considering in full generality the case of spacetimes admitting an abelian orthogonally transitive G_2 with spacelike orbits and one proper conformal Killing vector[1] such that the three-dimensional conformal group has either timelike or spacelike orbits (for null orbits see [19]). The final restriction we make is considering only the three-dimensional conformal Lie algebras for which the abelian isometry subalgebra is an ideal (i.e. the Lie bracket of the CKV with any of the two Killing vectors is a linear combination of only the two Killing vectors). We will not assume that the metric is diagonal (thus, we will study both classes B(i) and B(ii) in Wainwright's classification [20]). Obviously, we will assume $b \neq 0$ in expression (13) below when restricting ourselves to the diagonal metrics. Although the assumption of two Killing vectors (KVs) and one proper CKV is the same as in [16], [17], in which the stationary and axisymmetric case was analyzed, the structure of the Lie algebras and differential equations to be solved turn out to be rather different.

The plan of the paper is as follows. In section 2 we write down the Einstein field equations in non-comoving coordinates when the spacetime admits an abelian G_2 acting orthogonally transitively on spacelike orbits. Then, we establish the necessary and sufficient conditions for a solution of the system of Einstein equations to represent a perfect fluid. We finish the section by showing that a new metric obtained by interchanging the coordinates t and x in all the metric functions of a given solution of the Einstein field equations has still the energy-momentum tensor with the structure of perfect fluid but with the 4-velocity switching from timelike to tachyonic and vice versa. This result allows us to simplify the work considerably as only one half of the cases need to be considered. In section 3 we write down the inequivalent three-dimensional conformal Lie algebras for which the isometry abelian subalgebra is an ideal. We then give the canonical line-elements for each inequivalent case (some of the forms of the Lie algebras

[1]Obviously, the conformal Killing vector is not unique since it is defined up to linear combinations of the Killing vectors and a non-vanishing constant factor. Throughout the paper we will refer to this equivalence class as 'the conformal Killing vector'.

and the line-elements are different to those in [3] although they are completely equivalent). In section 4 we study the non-diagonal metrics and proof that *no* perfect-fluid solutions with this specification are possible. Thus, in section 5 we concentrate on the diagonal metrics (when $b \neq 0$, see above) and find the general solution of Einstein's field equations. There exist four families of solutions (each of them containing arbitrary parameters). These families are written explicitly in a simplified form in section 6. Readers interested mainly in the resulting metrics may advance to this section where we give the Petrov type of the spacetimes, the form of the fluid velocity vector, the shear tensor components, the acceleration and the expansion of the fluid (vanishing rotation is already a consequence of imposing an orthogonally transitive G_2). The expressions for the energy-density and pressure of the fluid are written down and we find for which values of the parameters the energy conditions are fulfilled. We also discuss the existence of an equation of state $p = p(\rho)$, the singularity structure of the spacetime and the FLRW limit cases for each of the families.

2 Some results on general abelian G_2 orthogonally transitive perfect fluids

As stated in the introduction, we are interested in cosmological abelian orthogonally transitive G_2 perfect-fluid solutions of Einstein's field equations. It is well-known that for such spacetimes there exist coordinates $\{t, x, y, z\}$ adapted to the Killing vectors in which the metric takes the form

$$ds^2 = \frac{1}{S^2(t,x)} \Big[-dt^2 + dx^2 + F(t,x) \left(P^{-1}(t,x) dy^2 + P(t,x) \left(dz + W(t,x) dy\right)^2 \right) \Big]. \quad (1)$$

The two Killing vectors are obviously given by

$$\vec{\xi} = \partial_y, \qquad \vec{\eta} = \partial_z.$$

We are going to analyze the Einstein field equations for the metric (1) when the matter content corresponds to a perfect fluid. Thus, the energy-momentum tensor takes the standard form

$$T_{\alpha\beta} = (\rho + p)\, u_\alpha u_\beta + p g_{\alpha\beta},$$

where ρ is the energy density of the fluid, p is the pressure and u_α is the fluid 4-velocity. For orthogonally transitive G_2 on S_2 spacetimes, the fluid velocity vector is necessarily orthogonal at each point to the group orbits. Thus, the one-form $\boldsymbol{u}$ is always a linear combination of the coordinate one-forms $\boldsymbol{dt}$ and $\boldsymbol{dx}$ at each point and, consequently, the fluid flow is irrotational. In all the cases below, we will write the Einstein field

equations using orthonormal tetrads $\{\boldsymbol{\theta}^{\boldsymbol{\alpha}}\}$ such that

$$\boldsymbol{\theta}^{\mathbf{0}} = \frac{1}{S(t,x)}\boldsymbol{dt}, \qquad \boldsymbol{\theta}^{\mathbf{1}} = \frac{1}{S(t,x)}\boldsymbol{dx}.$$

Therefore, the fluid one-form will always take the form

$$\boldsymbol{u} = u_0\boldsymbol{\theta}^{\mathbf{0}} + u_1\boldsymbol{\theta}^{\mathbf{1}}, \qquad u_0^2 - u_1^2 = 1, \tag{2}$$

so that the Einstein field equations, in units where $c = 8\pi G = 1$, read

$$\begin{gathered} S_{00} = (\rho + p)\, u_0^2 - p, \qquad S_{01} = (\rho + p)\, u_0 u_1, \qquad S_{11} = (\rho + p)\, u_1^2 + p, \\ S_{22} = S_{33} = p, \qquad S_{02} = S_{03} = S_{12} = S_{13} = S_{23} = 0, \end{gathered}$$

where $S_{\alpha\beta}$ stands for the components of the Einstein tensor in the $\{\boldsymbol{\theta}^{\boldsymbol{\alpha}}\}$ cobasis.

Given that we are not assuming any particular equation of state for the fluid, these Einstein field equations can be rewritten in terms of the Einstein tensor only. The energy density, the pressure and the form of the fluid velocity vector can be found afterwards, once the field equations are solved. Due to the separation into two orthogonal blocks of the metric, some of the Einstein tensor components are identically vanishing. In fact, a direct calculation shows that S_{02}, S_{03}, S_{12} and S_{13} vanish and consequently, the only equations we need to consider are

$$(S_{00} + S_{22})\,(S_{11} - S_{22}) - S_{01}^2 = 0, \tag{3}$$

$$S_{22} - S_{33} = 0, \tag{4}$$

$$S_{23} = 0. \tag{5}$$

Equations (3)-(5) are satisfied by all perfect-fluid metrics (with an orthogonally transitive G_2) but not all the solutions of this system represent a perfect-fluid spacetime. Let us find which are the restrictions we need to impose to obtain the perfect-fluid solutions (in general these conditions can only be checked once the equations are solved). The quadratic equation (3) corresponds exactly to the vanishing of the determinant of the 2×2 matrix

$$\begin{pmatrix} S_{00} + S_{22} & S_{01} \\ S_{01} & S_{11} - S_{22} \end{pmatrix},$$

which is clearly equivalent to the existence of two functions U_0 and U_1 such that

$$S_{00} + S_{22} = \epsilon U_0^2, \qquad S_{01} = \epsilon U_0 U_1, \qquad S_{11} - S_{22} = \epsilon U_1^2,$$

where ϵ is an appropriate sign. Only when the two functions U_0 and U_1 satisfy

$$U_0^2 - U_1^2 > 0$$

we can define the two components u_0 and u_1 of the fluid velocity vector through the expressions

$$\epsilon U_0^2 \equiv (\rho + p)\, u_0^2, \qquad \epsilon U_1^2 \equiv (\rho + p)\, u_1^2, \qquad \text{with} \qquad u_0^2 - u_1^2 = 1,$$

(which also define $\rho + p$ as $\rho + p = S_{00} - S_{11} + 2S_{22}$). Obviously, for physically well-behaved perfect fluids ϵ must be positive so that $\rho + p > 0$, but this is not necessary from a mathematical point of view for a solution of (3)-(5) to be a perfect fluid. The separate expression for the density and pressure are obtained from $S_{22} = S_{33} = p$. However, the solutions of (3)-(5) can also satisfy $U_0^2 - U_1^2 \leq 0$. When

$$U_0^2 - U_1^2 = 0 \iff S_{00} + S_{22} = S_{11} - S_{22}$$

the Einstein tensor (and therefore the energy-momentum tensor) takes the form

$$S_{\alpha\beta} = K_\alpha K_\beta + p g_{\alpha\beta}$$

where $\vec{K}$ is a null vector lying in the two plane spanned by the vectors ∂_t and ∂_x. This kind of solution represents a radiative fluid with pressure (which should satisfy $p = 0$ to be physically reasonable). As we are looking for perfect-fluid cosmologies we will not be interested in these radiative solutions here.

Finally, when $U_0^2 - U_1^2 < 0$, the energy-momentum corresponds to a perfect fluid but with the "fluid" velocity being spacelike. We will obviously discard this case in the following.

To summarize, the conditions that the solutions of the system (3)-(5) should satisfy in order to represent a perfect fluid can be written as

$$S_{00} - S_{11} + 2S_{22} \neq 0, \qquad \frac{S_{00} + S_{22}}{S_{00} - S_{11} + 2S_{22}} > 0. \tag{6}$$

Our next aim will be to impose the existence of a proper CKV in the spacetime (one of the main assumptions in our work). Before entering into this we want to state an interesting property for general abelian orthogonally transitive G_2 on S_2 spacetimes which will simplify considerably our work.

Lemma 1 *If the spacetime*

$$ds^2 = \frac{1}{S^2(t,x)} \left[-dt^2 + dx^2 + F(t,x) \left(P^{-1}(t,x) dy^2 + P(t,x) \left(dz + W(t,x) dy \right)^2 \right) \right] \tag{7}$$

satisfies the equations (3), (4) and (5), then the new metric

$$ds^2 = \frac{1}{S^2(x,t)} \left[-dt^2 + dx^2 + F(x,t) \left(P^{-1}(x,t) dy^2 + P(x,t) \left(dz + W(x,t) dy \right)^2 \right) \right] \tag{8}$$

obtained by interchanging x and t in all metric functions, also satisfies the set of differential equations (3), (4) and (5).

It is worth pointing out (and we will see that explicitly in the proof of the lemma) that if the first spacetime represents a perfect fluid then the second one is *not* a perfect-fluid solution because the conditions (6) are not satisfied (the sign of u^2 is switched). Thus, this lemma does not allow us to find new perfect-fluid solutions from old ones. The interest of the result lies in the fact that the amount of work to be done is reduced considerably in many problems. In particular, we will see when imposing the existence of a CKV that we need to consider only one half of all different possibilities. Let us now prove the lemma.

Proof: In order to avoid confusion, we will write the second metric (8) using coordinates T and X instead of t and x. So the metric reads

$$ds^2 = \frac{1}{S^2(X,T)}\left[-dT^2 + dX^2 + F(X,T)\left(P^{-1}(X,T)dy^2 + P(X,T)\left(dz + W(X,T)dy\right)^2\right)\right]. \tag{9}$$

Let us now consider the following line-element

$$ds^2 = \frac{1}{S^2(t,x)}\left[-\sigma dt^2 + \sigma dx^2 + F(t,x)\left(P^{-1}(t,x)dy^2 + P(t,x)\left(dz + W(t,x)dy\right)^2\right)\right], \tag{10}$$

where $\sigma = \pm 1$. It is clear that when $\sigma = 1$ this metric is exactly the same as (7) while when $\sigma = -1$ the coordinate change

$$t = X, \qquad x = T$$

transforms the metric into (9). We are now going to analyze the Einstein tensor for the metric (10). We choose the tetrad

$$\boldsymbol{\theta^0} = \frac{1}{S(t,x)}\boldsymbol{dt}, \quad \boldsymbol{\theta^1} = \frac{1}{S(t,x)}\boldsymbol{dx},$$
$$\boldsymbol{\theta^2} = \frac{1}{S(t,x)}\sqrt{\frac{F(t,x)}{P(t,x)}}\boldsymbol{dy}, \quad \boldsymbol{\theta^3} = \frac{1}{S(t,x)}\sqrt{F(t,x)P(t,x)}\left(\boldsymbol{dz} + W(t,x)\boldsymbol{dy}\right). \tag{11}$$

The scalar products of these tetrad one-forms are, obviously,

$$\left(\boldsymbol{\theta^\alpha}, \boldsymbol{\theta^\beta}\right) = \mathrm{diag}\left(-\sigma, \sigma, 1, 1\right).$$

The calculation of the Einstein tensor using this tetrad shows that in changing $\sigma \to -\sigma$ the components S_{22}, S_{23}, S_{33} switch sign whereas S_{00}, S_{01}, S_{11} remain unchanged (all the rest of components of the Einstein tensor are zero in both cases $\sigma = 1$ and $\sigma = -1$). We must now compare the components of the Einstein tensor of (9) with the components of the Einstein tensor of (7) when in both cases an orthonormal tetrad with timelike $\boldsymbol{\theta^0}$ is used (because this was used in (2) in order to write Einstein's equations in the

form (3)-(5)). When $\sigma = 1$ the tetrad (11) has already $\boldsymbol{\theta^0}$ timelike but when $\sigma = -1$ we must interchange the superscripts 0 to 1 because x is now the timelike coordinate. So, we find that the relationship between the Einstein tensors of the metrics (7) and (8) using respectively orthonormal tetrads with $\boldsymbol{\theta^0}$ being timelike is given by

$$\begin{array}{ccc} S^{(2)}_{00} = S^{(1)}_{11}, & S^{(2)}_{01} = S^{(1)}_{01}, & S^{(2)}_{11} = S^{(1)}_{00}, \\ S^{(2)}_{22} = -S^{(1)}_{22}, & S^{(2)}_{23} = -S^{(1)}_{23}, & S^{(2)}_{33} = -S^{(1)}_{33}, \end{array}$$

where the superscript (1) denotes the first metric (7) and the superscript (2) denotes the second metric (9). Consequently, if $S^{(1)}_{\alpha\beta}$ satisfies the equations (3), (4) and (5) then $S^{(2)}_{\alpha\beta}$ also satisfies the same equations and the lemma is proven. Recalling that the conditions for the solutions of these equations to represent perfect fluids are (6) and given that

$$S^{(2)}_{00} + S^{(2)}_{22} = S^{(1)}_{11} - S^{(1)}_{22}, \qquad S^{(2)}_{11} - S^{(2)}_{22} = S^{(1)}_{00} + S^{(1)}_{22},$$

we find that in spacetime regions where one metric represents a perfect fluid, the other represents a tachyon fluid and vice versa.

□

Until now the discussion has dealt with general abelian G_2 orthogonally transitive perfect fluids. In the next section we will impose the existence of a CKV, classify the inequivalent Lie algebras and write down canonical line-elements for each resulting case.

3 Inequivalent Lie algebras and line-elements

The main assumption in this paper is the existence of a proper conformal motion in the manifold. We do not assume any further conformal symmetries in the manifold, but we assume that the two abelian KVs and the CKV, $\vec{k}$, span a three-dimensional conformal Lie algebra which, in general, satisfies

$$\left[\vec{\xi}, \vec{\eta}\right] = \vec{0}, \quad \left[\vec{\xi}, \vec{k}\right] = \gamma_1\vec{\xi} + \gamma_2\vec{\eta} + \gamma_3\vec{k}, \quad \left[\vec{\eta}, \vec{k}\right] = \delta_1\vec{\xi} + \delta_2\vec{\eta} + \delta_3\vec{k}, \tag{12}$$

where all γ's and δ's are obviously constants (in the presence of a fourth symmetry the existence of the three-dimensional Lie subalgebra (12) could be violated). As a short calculation shows, all Lie algebras (12) are solvable (see e.g. [18] for a definition of solvability). We in the following make a stronger assumption. We assume that the two-dimensional abelian subalgebra generated by the two KVs in (12) is an *ideal* (understanding the multiplication in the algebra as taking the commutator), i.e. we assume $\gamma_3 = \delta_3 = 0$. The line-element and consequently the Einstein field equations take very different forms whether the isometry subalgebra is an ideal or not. Thus,

the two cases must be treated separately. In this paper we will concentrate only on the case when the isometry subalgebra is an ideal. *Thus, we assume from now on that* $\gamma_3 = \delta_3 = 0$. We must now find canonical forms for the inequivalent Lie algebras contained in (12) for $\gamma_3 = \delta_3 = 0$. In order to do so, we can exploit the freedom in performing linear transformations of the KVs $\vec{\xi}$ and $\vec{\eta}$. Under these transformations, the 2×2 matrix

$$\left(\begin{array}{cc} \gamma_1 & \gamma_2 \\ \delta_1 & \delta_2 \end{array} \right)$$

transforms like an endomorphism. By doing this, we can reach a canonical Jordan form for this matrix. Three different possibilities arise depending on whether the matrix diagonalizes with real eigenvalues (Case A), diagonalizes with complex eigenvalues (Case VII) or does not diagonalize (Case B). We used the term VII because that Lie algebra corresponds to the Bianchi type VII in Bianchi's classification of three-dimensional Lie algebras. Each one of the other two cases, A and B, contains several different Bianchi types and no similar Bianchi type name can be used (I,III,V,VI are in A and II,IV are in B). In the complex case, we then perform a transformation to obtain a real valued matrix. The three possibilities read, finally,

Lie Algebra A

$$\left[\vec{\xi},\vec{\eta}\right] = \vec{0}, \quad \left[\vec{\xi},\vec{k}\right] = \frac{1}{2}\left(c+b\right)\vec{\xi}, \quad \left[\vec{\eta},\vec{k}\right] = \frac{1}{2}\left(c-b\right)\vec{\eta}, \tag{13}$$

where b and c are arbitrary (possibly vanishing) constants.

Lie Algebra B

$$\left[\vec{\xi},\vec{\eta}\right] = \vec{0}, \quad \left[\vec{\xi},\vec{k}\right] = \frac{1}{2}c\vec{\xi} + a\vec{\eta}, \quad \left[\vec{\eta},\vec{k}\right] = \frac{1}{2}c\vec{\eta},$$

where a is a non-vanishing constant.

Lie Algebra VII

$$\left[\vec{\xi},\vec{\eta}\right] = \vec{0}, \quad \left[\vec{\xi},\vec{k}\right] = \frac{1}{2}c\vec{\xi} - a\vec{\eta}, \quad \left[\vec{\eta},\vec{k}\right] = a\vec{\xi} + \frac{1}{2}c\vec{\eta}.$$

where a and c are constants ($a \neq 0$).

In order to find the line-elements for each of these Lie algebras we must solve the conformal Killing equations

$$\mathcal{L}_{\vec{k}}g_{\alpha\beta} = 2\Phi g_{\alpha\beta},$$

where $\mathcal{L}_{\vec{k}}$ represents the Lie derivative along the vector field $\vec{k}$ and Φ is a scalar function usually called scale factor. Given that we will concentrate on spacetimes admitting a *proper* CKV, the scale factor will be assumed to be non-constant (and in particular not identically zero). When solving the conformal Killing equations, it is found that three different cases must be considered depending on the character (spacelike, timelike or null) of the orbits of the three-dimensional conformal group (which is generated by the Lie algebra $\{\vec{\xi},\vec{\eta},\vec{k}\}$). In this paper we will not study the case when these orbits are null and, instead, we will concentrate on the cases when the orbits are either timelike or spacelike. Let us first write down explicitly the canonical line-elements for each of the Lie algebras A,B and VII when the orbits of the conformal group are timelike.

Lie Algebra A

$$ds^2 = \frac{1}{S^2(t,x)}\Big[-dt^2 + dx^2 + F(x)P^{-1}(x)e^{-(b+c)t}dy^2 + \\ + F(x)P(x)e^{(b-c)t}\left(dz + W(x)e^{-bt}dy\right)^2\Big]. \quad (14)$$

The expression for the conformal Killing vector in this metric is

$$\vec{k} = \partial_t + \frac{(c+b)}{2}y\partial_y + \frac{(c-b)}{2}z\partial_z.$$

The orthonormal tetrad we will use for this Lie algebra is

$$\boldsymbol{\theta^0} = \frac{1}{S}\boldsymbol{dt},\ \ \boldsymbol{\theta^1} = \frac{1}{S}\boldsymbol{dx},\ \ \boldsymbol{\theta^2} = \frac{1}{S}\sqrt{\frac{F}{P}}e^{-\frac{b+c}{2}t}\boldsymbol{dy},\ \ \boldsymbol{\theta^3} = \frac{\sqrt{FP}}{S}e^{\frac{b-c}{2}t}\left(\boldsymbol{dz} + We^{-bt}\boldsymbol{dy}\right). \quad (15)$$

Lie Algebra B

$$ds^2 = \frac{1}{S^2(t,x)}\Big[-dt^2 + dx^2 + \\ + F(x)e^{-ct}\left(P^{-1}(x)dy^2 + P(x)\left(dz + [W(x)+at]\,dy\right)^2\right)\Big]. \quad (16)$$

The conformal Killing vector in this metric reads

$$\vec{k} = \partial_t + \frac{1}{2}cy\partial_y + \left(ay + \frac{1}{2}cz\right)\partial_z.$$

The orthonormal tetrad we will use is

$$\boldsymbol{\theta^0} = \frac{1}{S}\boldsymbol{dt}, \quad \boldsymbol{\theta^1} = \frac{1}{S}\boldsymbol{dx},$$
$$\boldsymbol{\theta^2} = \frac{1}{S}\sqrt{\frac{F}{P}}e^{-\frac{c}{2}t}\boldsymbol{dy}, \quad \boldsymbol{\theta^3} = \frac{\sqrt{FP}}{S}e^{-\frac{c}{2}t}\left(\boldsymbol{dz} + [W+at]\,\boldsymbol{dy}\right). \quad (17)$$

Lie Algebra VII

In this case the explicit form of the metric is much longer than before. We will give, instead, the explicit form of the orthonormal tetrad we will use in our calculations. It reads

$$\boldsymbol{\theta^0} = \frac{1}{S(t,x)}\boldsymbol{dt}, \quad \boldsymbol{\theta^1} = \frac{1}{S(t,x)}\boldsymbol{dx},$$

$$\boldsymbol{\theta^2} = \frac{1}{S(t,x)}\sqrt{\frac{F(x)}{P(x)}}e^{-\frac{c}{2}t}\left[\cos(at)\,\boldsymbol{dy} - \sin(at)\,\boldsymbol{dz}\right], \tag{18}$$

$$\boldsymbol{\theta^3} = \frac{\sqrt{F(x)P(x)}e^{-\frac{c}{2}t}}{S(x,t)}\Big(\ [\cos(at) - W(x)\sin(at)]\,\boldsymbol{dz} + [\sin(at) + W(x)\cos(at)]\,\boldsymbol{dy}\Big).$$

The conformal Killing vector in this metric is

$$\vec{k} = \partial_t + \left(\frac{1}{2}cy + az\right)\partial_y + \left(-ay + \frac{1}{2}cz\right)\partial_z.$$

The scale factor of the CKV for each one of these metrics reads

$$\Phi = -\frac{S_{,t}}{S}. \tag{19}$$

For the remaining cases in which the orbits of the conformal group are spacelike, the line-elements can be obtained directly from the previous ones by simply interchanging t and x in the metric coefficients as described in section 2. It is now obvious that making use of the lemma 1 we can restrict the study to the case when the orbits of the conformal group are timelike because in finding the general solution of the three equations (3), (4) and (5) we will also find the general solution for the metrics obtained by interchanging the roles of x and t in the metric coefficients, which correspond exactly to the metrics when the orbits of the three-dimensional conformal group are spacelike.

In the following section we will concentrate on the non-diagonal metrics and will prove that there do not exist perfect-fluid solutions under the assumptions of this paper when the line-element is non-diagonal.

4 Non-diagonal metrics

In this section we are going to prove that all the perfect-fluid solutions of Einstein field equations for the spacetimes we are considering in this paper (see beginning of section 3 for a clear account of our assumptions) must be diagonal. In the previous section we found the canonical line-elements for the three possible Lie algebras (A, B and VII). It is clear from the expressions for these line-elements that all the metrics belonging to Lie algebras B and VII must be necessarily non-diagonal (irrespective of any field

equations) and that only for Lie algebra A the diagonal case is possible (setting $W = 0$ in (14)). Thus, in order to prove that no non-diagonal perfect fluids exist, we must show that the metrics in Lie algebras B and VII do not admit perfect-fluid solutions at all and that only the diagonal metrics in Lie algebra A admit perfect-fluid solutions. These results will be stated in the following three propositions (each one corresponding to one of the Lie algebras A,B and VII). Let us start with the simplest case, namely the one corresponding to Lie algebra B.

Proposition 1 *No perfect-fluid solutions exist for the conformal Lie Algebra B with $\vec{k}$ being a proper CKV.*

Proof: We will concentrate on the two Einstein field equations (4) and (5) which, using the tetrad (17), read respectively

$$\frac{P'}{P}\frac{F'}{F} + \frac{P''}{P} - \frac{P'^2}{P^2} - W'^2 P^2 + a^2 P^2 - 2\frac{P'}{P}\frac{S_{,x}}{S} = 0 \tag{20}$$

$$W'\left(\frac{1}{2}\frac{F'}{F} + \frac{P'}{P} - \frac{S_{,x}}{S}\right) + \frac{1}{2}W'' + \frac{1}{2}ac + a\frac{S_{,t}}{S} = 0 \tag{21}$$

where the prime denotes derivative with respect to x and the comma indicates partial derivative. Let us first consider the situation in which $P' \neq 0$. The first equation above shows that the function S must take the form of a product of a function of x and a function of t, $S = G(x)H(t)$. Then, the second equation reads

$$W'\left(\frac{1}{2}\frac{F'}{F} + \frac{P'}{P} - \frac{G'}{G}\right) + \frac{1}{2}W'' + \frac{1}{2}ac + a\frac{\dot{H}}{H} = 0,$$

which immediately implies that $\dot{H}/H$ must be a constant (remember that for the Lie algebra B the constant a cannot vanish) and then the expression for the scale factor (19) implies that the CKV is in fact homothetic. Thus no perfect fluid with a proper conformal motion does exist in this case. It only remains to analyze the situation when P satisfies $P' \equiv 0$. The equation (20) gives simply

$$W' = \epsilon_1 a,$$

where $\epsilon_1 = \pm 1$. The second equation (21) takes the form

$$\epsilon_1 \frac{1}{2}\frac{F'}{F} + \frac{1}{2}c + \frac{S_{,t}}{S} - \epsilon_1 \frac{S_{,x}}{S} = 0,$$

which can be integrated to give

$$S = \sqrt{F}\exp\left(\epsilon_1 \frac{c}{2} x\right) U\left(t + \epsilon_1 x\right)$$

with U an arbitrary function of $t + \epsilon x$. The quadratic Einstein equation (3) takes now the very simple form

$$\left(\frac{F''}{F} - \frac{F'^2}{F^2}\right)^2 = 0 \qquad \Longleftrightarrow \qquad F = F_0 e^{\beta x},$$

with constants F_0 and β. Now we can evaluate the components of the Einstein tensor to find

$$S_{00} = S_{11} = \epsilon_1 S_{01}, \qquad\qquad S_{22} = S_{33} = S_{23} = 0,$$

which shows that the energy-momentum content is that of a null fluid without pressure. Thus, we can conclude that no perfect-fluid solutions with a proper CKV exist for the Lie algebra of the type B. Let us emphasize once more that this analysis covers both cases, when the orbits of the conformal group are timelike and spacelike.

Let us now prove a similar result for Lie algebra VII, i.e. that no perfect-fluid solutions arise for Lie algebra VII.

Proposition 2 *No perfect-fluid solutions of Einstein's field equations do exist for space-times*

- *admitting a three-dimensional Lie algebra of class VII of two KVs and one proper CKV with*

- *the two-dimensional isometry group being maximal.*

Proof: The proof for this case is similar to the previous one and we will only outline the main steps. We concentrate, as before, on the two Einstein equations (4) and (5) which, using the tetrad (18), read respectively

$$\begin{aligned}\frac{F'}{F}\frac{P'}{P} + \left(\frac{P'}{P}\right)' - 2\frac{P'}{P}\frac{S_{,x}}{S} - 4aW\frac{S_{,t}}{S} - \\ -W'^2P^2 + a^2P^2\left(W^2+1\right)^2 - \frac{a^2}{P^2} - 2acW = 0\end{aligned} \tag{22}$$

$$\begin{aligned}W'\left(\frac{1}{2}\frac{F'}{F} + \frac{P'}{P} - \frac{S_{,x}}{S}\right) + a\left(\frac{S_{,t}}{S} + \frac{c}{2}\right)\left(W^2 + 1 - \frac{1}{P^2}\right) + \\ +\frac{1}{2}W'' + a^2W\left(W^2 + 1 + \frac{1}{P^2}\right) = 0.\end{aligned} \tag{23}$$

We must again distinguish between two cases depending on whether $P' \equiv 0$ or not. When $P = \text{const}$, the only non-homothetic solution of the system (22)-(23) is $P = 1$ and $W = 0$. Then, the tetrad reduces to

$$\boldsymbol{\theta^0} = \frac{\boldsymbol{dt}}{S}, \quad \boldsymbol{\theta^1} = \frac{\boldsymbol{dx}}{S}, \quad \boldsymbol{\theta^3} = \frac{1}{S}\sqrt{F}e^{-\frac{c}{2}t}\left(\cos(at)\,\boldsymbol{dy} - \sin(at)\,\boldsymbol{dz}\right),$$
$$\boldsymbol{\theta^4} = \frac{1}{S}\sqrt{F}e^{-\frac{c}{2}t}\left(\sin(at)\,\boldsymbol{dy} + \cos(at)\,\boldsymbol{dz}\right)$$

and the line-element is simply

$$ds^2 = \frac{1}{S(x,t)^2}\left[-dt^2 + dx^2 + F(x)e^{-ct}\left(dy^2 + dz^2\right)\right]$$

which has a three-dimensional isometry group (against our assumptions). Thus, we must consider the case $P' \neq 0$. It is not difficult to see that the only possibility for proper conformal solutions is

$$S = G(x)H(t + aK(x)), \quad \left(\text{with} \quad K' = -2W\frac{P}{P'}\right)$$

$$\frac{P'}{P} = 2\epsilon_2 aW, \qquad \frac{G'}{G} = \frac{1}{2}\left(\frac{F'}{F} - \epsilon_2 c\right), \qquad W^2 = \frac{m^2}{P} - 1 - \frac{1}{P^2},$$

where m is a constant, $\epsilon_2 = \pm 1$ and H is an arbitrary function of its argument. Imposing now the quadratic Einstein equation (3) we simply find

$$\left(\frac{F''}{F} - \frac{F'^2}{F^2}\right)^2 = 0 \qquad \Longleftrightarrow \qquad F = F_0 e^{\beta x},$$

where F_0 and β are constants of integration. As in the previous case, the evaluation of the Einstein tensor for this solution gives us

$$S_{00} = S_{11} = -\epsilon_2 S_{01}, \qquad S_{22} = S_{33} = S_{23} = 0$$

so that the matter content is a null fluid without pressure and the proof of the proposition is completed.

Only Lie algebra A remains to be analyzed. Let us prove that all the perfect-fluid solutions with a proper CKV belonging to Lie algebra A must be diagonal.

Proposition 3 *No perfect-fluid solutions of Einsteins field equations do exist for space-times*

- *admitting a three-dimensional Lie algebra of class A of two Killing vectors and one proper CKV with*
- *the maximal two-dimensional isometry group being maximal and*
- *the metric being non-diagonal.*

The proof is again similar and we will just sketch it. The two equations (4) and (5) read respectively

$$\frac{P'}{P}\frac{F'}{F} + \left(\frac{P'}{P}\right)' - 2\frac{P'}{P}\frac{S_{,x}}{S} + 2b\frac{S_{,t}}{S} - W'^2 P^2 + b^2 P^2 W^2 + bc = 0, \tag{24}$$

$$W'\left(\frac{1}{2}\frac{F'}{F} + \frac{P'}{P} - \frac{S_{,x}}{S}\right) + \frac{1}{2}W'' + \frac{1}{2}b\,(b-c)\,W - bW\frac{S_{,t}}{S} = 0. \tag{25}$$

The $P' \equiv 0$ subcase gives rise either to homothetic spacetimes or to metrics with a three-dimensional group of isometries acting on two-dimensional spacelike surfaces. Thus, we can move to the general case $P' \neq 0$. Equation (24) gives

$$S = G(x)H(t + bK(x)) \qquad \text{with} \quad K' = \frac{P}{P'}.$$

which inserted into (25) produces

$$W'\left(\frac{1}{2}\frac{F'}{F} + \frac{P'}{P} - \frac{G'}{G}\right) + \frac{1}{2}W'' + \frac{1}{2}b\,(b-c)\,W = \frac{\dot{H}}{H}b\left(W + W'\frac{P}{P'}\right).$$

The conditions that the CKV is proper implies that both sides in this equation must vanish. In particular, the right-hand side implies that either $b = 0$ or $W = \alpha/P$ (where α is an arbitrary constant). Let us start with $b = 0$. The two equations (24) and (25) take now the form

$$\frac{P'}{P}\left(\frac{F'}{F} - 2\frac{G'}{G}\right) + \left(\frac{P'}{P}\right)' - W'^2P^2 = 0, \tag{26}$$

$$W'\left(\frac{1}{2}\frac{F'}{F} - \frac{G'}{G} + \frac{P'}{P}\right) + \frac{1}{2}W'' = 0 \tag{27}$$

and the second one can be integrated to give

$$W' = W_0\frac{G^2}{FP^2},$$

where W_0 is an arbitrary constant. The case $W_0 = 0$ gives a diagonal metric and, therefore, we can assume $W_0 \neq 0$. Then, equation (26) can be integrated to give

$$\frac{1}{P^2} = \delta^2 - (W + \beta)^2 \tag{28}$$

(where β and δ are arbitrary constants of integration). The metric is apparently non-diagonal, but a trivial calculation using (28) shows that the linear coordinate change in the block $\{y, z\}$

$$y = Y + Z, \quad z = (\beta + \delta)\,Y + (\beta - \delta)\,Z$$

brings the metric into a diagonal form and therefore no proper non-diagonal solutions do exist in this subcase (notice that this coordinate change is singular only when $\delta = 0$ which is impossible due to (28)).

We can now consider that last possibility, namely, when $b \neq 0$ and $W = \alpha/P$ (with $\alpha \neq 0$ in order to consider non-diagonal metrics). Substituting W everywhere, it is easy

to see that the general solution of the full set of Einstein field equations (3), (4) and (5) is

$$\frac{P'}{P} = \sigma b, \qquad \frac{G'}{G} = \frac{1}{2}\left(n + \sigma c\right), \qquad \frac{F'}{F} = n,$$

where $\sigma = \pm 1$ and n is an arbitrary constant. As in the two previous cases this solution corresponds to a null fluid and the proof of the proposition is completed.

To summarize, we have proven in this section that there do not exist any non-diagonal perfect-fluid solutions for spacetimes with

- a maximal abelian orthogonally transitive G_2 on S_2
- admitting one proper CKV (with conformal orbits either spacelike or timelike) such that
- the subalgebra generated by the two Killing vectors is an ideal in the Lie algebra spanned by the two KVs and the CKV.

In the following section we will concentrate on the diagonal case and find all the perfect-fluid solutions (with $b \neq 0$, see the introduction).

5 Diagonal perfect-fluid solutions

We know that only the Lie algebra A admits diagonal metrics (all metrics in Lie algebras B and VII are necessarily non-diagonal and, as we have seen in the previous section, none of them satisfies the perfect-fluid Einstein field equations). In order to restrict the metrics for Lie algebra A to be diagonal we must set $W \equiv 0$ in (14) (and also in the corresponding tetrad (15)). Now, the calculation of the Einstein tensor in this tetrad gives $S_{23} \equiv 0$ so that the Einstein field equation (5) is identically satisfied (this can be seen directly from (25) after substituting $W = 0$). Equation (24) with $W = 0$ gives ($P =$ const. would imply $b = 0$ and a third KV):

$$S = G(x)H(t + bK(x)) \qquad \text{with} \qquad K' = \frac{P}{P'},$$
$$\frac{P'}{P}\left(\frac{F'}{F} - 2\frac{G'}{G}\right) + \left(\frac{P'}{P}\right)' + bc = 0. \tag{29}$$

Now, we must inevitably analyze the more difficult quadratic equation (3) which takes the form

$$\Sigma_1(x) + \frac{\dot{H}}{H}\Sigma_2(x) + \frac{\dot{H}^2}{H^2}\Sigma_3(x) + \frac{\ddot{H}}{H}\frac{\dot{H}}{H}\Sigma_4(x) + \frac{\ddot{H}}{H}\Sigma_5(x) = 0, \tag{30}$$

where $\Sigma_i(x)$ ($i = 1$ to 5) are expressions containing P, F and G and their derivatives and which depend only on the variable x. The dot means from now on derivative with respect to the variable $t + bK(x)$. In the Appendix we find all the different possibilities compatible with this mixed equation containing both independent variables x and $t + bK$. Let us here just summarize the results.

Case 1

$$\Sigma_1 = \Sigma_2 = \Sigma_3 = \Sigma_4 = \Sigma_5 = 0, \qquad H \quad \text{arbitrary}$$

Case 2

$$\Sigma_4 = 0, \quad \Sigma_1 = m_1\Sigma_5, \quad \Sigma_2 = m_2\Sigma_5, \quad \Sigma_3 = m_3\Sigma_5, \quad \Sigma_5 \neq 0$$
$$m_1 + m_2\frac{\dot{H}}{H} + m_3\frac{\dot{H}^2}{H^2} + \frac{\ddot{H}}{H} = 0,$$

where m_1, m_2 and m_3 are arbitrary constants.

Case 3

$$\Sigma_1 = k_1\Sigma_4, \quad \Sigma_2 = k_2\Sigma_4, \quad \Sigma_3 = k_3\Sigma_4, \quad \Sigma_5 = k_5\Sigma_4, \quad \Sigma_4 \neq 0$$
$$k_1 + k_2\frac{\dot{H}}{H} + k_3\frac{\dot{H}^2}{H^2} + \frac{\dot{H}}{H}\frac{\ddot{H}}{H} + k_5\frac{\ddot{H}}{H} = 0,$$

where k_1, k_2, k_3 and k_5 are arbitrary constants.

Case 4

$$\Sigma_1 = k_1\Sigma_5, \quad \Sigma_2 = k_1\Sigma_4 + k_3\Sigma_5, \quad \Sigma_3 = k_3\Sigma_4, \quad \Sigma_5 \neq 0, \quad \frac{\Sigma_4}{\Sigma_5} \neq \text{const.}$$
$$k_1 + k_3\frac{\dot{H}}{H} + \frac{\ddot{H}}{H} = 0$$

where k_1 and k_3 are arbitrary constants.

The explicit expressions for Σ_i will be necessary later to study these four different cases. They can be directly obtained from the following expressions.

$$\Sigma_1 \equiv Z_1V_1 - W_1^2, \qquad \Sigma_2 \equiv Z_1V_2 + Z_2V_1 - 2W_1W_2, \qquad \Sigma_3 \equiv Z_2V_2 - W_2^2,$$
$$\Sigma_4 \equiv -2\left(2W_2R + Z_2R^2 + V_2\right), \qquad \Sigma_5 \equiv -2\left(2W_1R + Z_1R^2 + V_1\right),$$

where Z_1, Z_2, V_1, V_2, W_1, W_2 and R are given explicitly in terms of the metric coefficients by

$$Z_1 \equiv \frac{1}{2}\frac{F''}{F} - \frac{F'}{F}\frac{G'}{G} + \frac{1}{2}b^2, \qquad Z_2 \equiv -c - b\frac{F'}{F}\frac{P}{P'},$$

$$V_1 \equiv \frac{1}{2}\frac{F''}{F} - \frac{1}{2}\frac{F'^2}{F^2} + \frac{F'}{F}\frac{G'}{G} - 2\frac{G''}{G} + \frac{1}{2}\frac{P'^2}{P^2} + \frac{1}{2}c^2, \qquad V_2 \equiv c - 2b^2 c\frac{P^2}{P'^2} - b\frac{F'}{F}\frac{P}{P'},$$

$$W_1 \equiv \frac{c}{2}\frac{F'}{F} - \frac{b}{2}\frac{P'}{P}, \qquad W_2 \equiv 2\frac{G'}{G}, \qquad R \equiv b\frac{P}{P'}.$$

As explained in the introduction, the case when the constant b in the Lie algebra A is identically zero has been exhaustively analyzed in [3]. Thus, we can restrict our study to the case when $b \neq 0$. We have to solve the four different sets of differential equations Case 1 to Case 4. Obviously, the two first cases are simpler because of the equation $\Sigma_4 = 0$. Let us first concentrate on these two cases. It is convenient to define three new functions $q(x)$, $l(x)$ and $m(x)$ through the relations

$$\frac{P'}{P} \equiv bq, \qquad \frac{F'}{F} \equiv (c+l)\,q, \qquad \frac{G'}{G} \equiv \frac{1}{q}\left(c + \frac{m}{4}\right).$$

The equation (29) reads now

$$q' + (c+l)\,q^2 - c - \frac{m}{2} = 0$$

which allows us to obtain m in terms of q and l. Using this expression for m, the equation $\Sigma_4 = 0$ takes the form

$$-2q' + (l+2c)\left(1 - q^2\right) = 0. \tag{31}$$

The solution $q \equiv \pm 1$ of this equation also implies $\Sigma_5 \equiv 0$ and, therefore, the only possibility is Case 1. The expressions for Σ_2 and Σ_3 are identically zero while the vanishing of Σ_1 imposes $l' = 0$. Now all the equations are satisfied. However, evaluating the Einstein tensor for this solution we find

$$S_{00} + S_{22} = S_{11} - S_{22} = \pm S_{01}, \qquad S_{22} = S_{33} = 0$$

so that it does not represent a perfect fluid. Thus, we can assume $q^2 \neq 1$ and obtain l from (31) as

$$l = -2c - \frac{2q'}{q^2 - 1}.$$

The expressions for Σ_5 and Σ_3 read now, respectively,

$$\Sigma_5 = -\frac{1}{q^2}\left[\left(q' + cq^2 - c\right)^2 + \left(b^2 - c^2\right)\left(q^2 - 1\right)^2\right],$$

$$\Sigma_3 = -\frac{1}{q^2}\left(q' + cq^2 - c\right)^2.$$

Let us start by analyzing Case 1. Given that both Σ_5 and Σ_3 must vanish we immediately find

$$q' = c\left(1-q^2\right), \qquad b = \epsilon c,$$

where ϵ is a sign. Now, all Σ_i are identically zero and we have another solution of the Einstein field equations. The calculation of the Einstein tensor shows that the matter content is indeed a perfect fluid. However, the spacetime is conformally flat and both the density and pressure depend only on the variable $t+bK$. Thus they satisfy an equation of state $p=p(\rho)$ and, consequently, the spacetime is a Friedman-Lemaître-Robertson-Walker cosmology. Consequently, Case 1 does not give new perfect-fluid solutions and we can move on to Case 2.

The equation $\Sigma_3 - m_3\Sigma_5 = 0$ reads

$$(1-m_3)\left(q'+cq^2-c\right)^2 = m_3\left(b^2-c^2\right)\left(q^2-1\right)^2. \tag{32}$$

When $m_3 \neq 1$ this equation immediately implies

$$q' = n\left(1-q^2\right), \tag{33}$$

where n is a constant. When $m_3=1$ the equation (32) implies $b=\epsilon c$ and then, combining the other two differential equations $\Sigma_1 - m_1\Sigma_5 = 0$ and $\Sigma_2 - m_2\Sigma = 0$ (two equations for one unknown $q(x)$), it is not difficult to show that the relation (33) must still hold. Substituting this expression for q' into all Σ_i we find that all the equations of Case 2 are fulfilled, and they simply give m_1, m_2 and m_3 in terms of the constants b, c and n. The explicit expressions are

$$m_3 = \frac{(c-n)^2}{n^2-2cn+b^2}, \qquad m_2 = \frac{(n-c)^3+3c\,(n-c)^2+c\,(b^2-c^2)}{n^2-2cn+b^2},$$
$$m_1 = \frac{1}{4}\frac{(c^2-n^2)\,(c^2-2cn+b^2)}{n^2-2cn+b^2}$$

and the metric functions are given by

$$\frac{P'}{P} = bq(x), \quad \frac{F'}{F} = (2n-c)\,q(x), \quad \frac{G'}{G} = \frac{1}{2}(n-c)\,q(x) + \frac{1}{2}\frac{n+c}{q(x)}, \quad q' = n\left(1-q^2\right),$$

where b, c and n are arbitrary constants. Let us label this solution as **Solution A**. We will write down the explicit line-element and analyze the physical content of this solution in the following section.

Thus, we have exhausted the Cases 1 and 2 and we have to deal now with the much more complicated Cases 3 and 4. We have to take advantage that in both cases the equation $\Sigma_3 - k_3\Sigma_4 = 0$ holds. Therefore, we will try to write this equation in the

simplest possible form so that the problem becomes more tractable. To that aim let us define three functions $r(x)$, $u(x)$ and $v(x)$ through the relations

$$\frac{F'}{F} = 2r(x) + u(x), \qquad \frac{G'}{G} = r(x), \qquad \frac{P'}{P} = \kappa b\sqrt{v(x)},$$

where κ is the sign of $P'/(bP)$ (so that the square root is non-negative definite, as usual). Now, equation (29) allows us to obtain u in terms of v as

$$u = -\frac{2c\kappa\sqrt{v} + v'}{2v}$$

and substitute it in all other expressions so that only the two functions $v(x)$ and $r(x)$ remain in the equations.

Let us consider the situation when v is a constant. Then we must assume $v \neq 1$ because otherwise we would have $\Sigma_4 \equiv 0$ and would be in a previous case. Now, the equation $\Sigma_3 - k_3\Sigma_4 = 0$ immediately implies $r = \text{const}$. So, we have

$$\frac{F'}{F} = 2a_1 - \frac{c}{a_2}, \qquad \frac{G'}{G} = a_1, \qquad \frac{P'}{P} = ba_2$$

(a_1 and a_2 constants) and we certainly have a solution of Case 3 (Case 4 is impossible in this situation because Σ_5 is constant and therefore proportional to Σ_4). The equations simply give k_1, k_2, k_3 and k_5 in terms of the constants c, b, a_1 and a_2. This is a solution of the Einstein field equations and its matter content is a perfect fluid, but it is trivial to checked that the metric contains a homothetic Killing vector (besides the proper CKV) and therefore does not belong to the class we are interested in this paper.

So, we can assume from now on that v is non-constant and perform the following change of variables

$$dv = 2\kappa\sqrt{v}\,(v-1)\,[\chi(v) - k_3]\,dx$$

so that now the independent variable is v and $\chi(v)$ is the unknown. Let us also define a function $\omega(v)$ through

$$r \equiv \frac{1}{2}\frac{\kappa}{\sqrt{v}}\,(\omega - \chi + 2k_3)$$

so that we replace r by ω everywhere. With all these changes and redefinitions, the common equation $\Sigma_3 - k_3\Sigma_4 = 0$ takes the simple form

$$v\left[\chi^2 - (c-k_3)^2\right] = \omega^2 - (c-k_3)^2\,. \tag{34}$$

This equation is linear in the independent variable v and, therefore, it would be convenient to consider χ as the independent variable and $v(\chi)$ the unknown. This change of

variables can be done unless χ is constant. Thus, we must consider the two cases $\chi =$ const. and χ non constant separately. Let us first assume $\chi =$ const.

Now, we have only one unknown $\omega(v)$ which must satisfy an overdetermined system of differential equations (four for Case 3 and three for Case 4). It is not difficult to find the solutions which are:

Case 3 and $\chi =$ const.

The solution in this case is

$$\chi = k_3 - \sigma_1 b, \qquad \omega = k_3 - \sigma_1 b, \qquad c = \sigma_1 b,$$
$$k_1 = -\frac{3b^2}{4}\left(\sigma_1 b - k_3\right), \qquad k_2 = \frac{b}{2}\left(-b + 4\sigma_1 k_3\right), \qquad k_5 = \frac{3}{2}\sigma_1 b,$$

where $\sigma_1 = \pm 1$ and b and k_3 are arbitrary constants. Let us label this solution as **Solution B** for further study in the next section.

Case 4 and $\chi =$ const.

Now, two different solutions are possible. The first one is

$$\chi = c - k_3, \qquad \omega = c - k_3, \qquad k_1 = 0, \qquad b^2 + c^2 - 2ck_3 = 0, \qquad b, c \text{ arbitrary}$$

which again will be studied in the following section. Let us call it **Solution C**. The other possibility reads

$$\omega = \sigma_2\left(c - \sigma_1 b\right)\sqrt{v}, \qquad p = c - \sigma_1 b, \qquad k_3 = c, \qquad k_1 = \frac{c^2 - b^2}{4},$$

where σ_2 is also a sign. This solution, however, can be seen to represent a null fluid (with non-zero pressure, in general) and, therefore, will not be analyzed any further.

Having written all the solutions of the equations when $\chi =$ const, let us consider the more general case when χ is non-constant. Now, as stated above, it is convenient to perform another change of variables and consider χ as the independent variable and v and ω as unknown functions depending on χ. Equation $\Sigma_3 - k_3\Sigma_4 = 0$, (34), allows us to obtain v in terms of χ as

$$v = \frac{\omega^2 - \left(c - k_3\right)^2}{\chi^2 - \left(c - k_3\right)^2}$$

and to substitute this expression in all the Σ_i. Now we have only one unknown $\omega(\chi)$ which must satisfy an overdetermined system of differential equations (three differential equations for Case 3 and two for Case 4). The investigation of the compatibility of

these systems of differential equations is now rather lengthy and makes essential use of computer algebra [13]. Special care is needed to combine the equations in the right order to avoid too long expressions. Let us here simply summarize which are the results obtained. It turns out that there exists one family of solutions for Case 3 and another family for Case 4. They are, explicitly,

Case 3 and χ non constant

$$\omega = \chi - 2\sigma_1 b + 2k_3, \qquad c = \sigma_1 b, \qquad k_1 = 0,$$
$$k_2 = \frac{b}{2}\left(2\sigma_1 k_3 - b\right), \qquad k_5 = k_3 - \frac{\sigma_1 b}{2}, \qquad b \text{ and } k_3 \text{ arbitrary.}$$

This solution will be labeled as **Solution D** and will be studied in the following section.

Case 4 and χ non constant

$$\omega = \chi\left(1 + \sigma_2\sqrt{\frac{k_3^2 - 4k_1 - b^2}{\left(\chi - k_3 + b\right)\left(\chi - k_3 - b\right)}}\right) \qquad k_1 \text{ and } k_3 \text{ arbitrary.}$$

The Einstein tensor for this solution satisfies $S_{00}+S_{22} = S_{11}-S_{22} = -S_{01}$ and, therefore, as discussed above, represents a null fluid. Consequently, it will not be analyzed any further.

To summarize, we have found four families of solutions which represent perfect fluids when the spacetime is assumed to have an abelian orthogonally transitive G_2 on S_2 and also one proper CKV which spans together with the two KVs a Lie algebra generating either timelike or spacelike orbits such that the isometry subalgebra is an ideal. These families have been labeled as Solution A, B, C and D respectively and will be studied in the next section.

6 Analysis of the Perfect-Fluid Solutions

The aim of this section is to write down the line-elements for the four solutions found in the previous section (Solutions A, B, C and D). In order to do so, we must integrate back some changes of variables which helped us to solve the Einstein field equations. We will not give the details as the calculations are not difficult and can be easily reproduced. In some cases we have also performed some redefinitions of constants and some coordinate changes in the block $\{t, x\}$ as well as rescalings of the coordinates y and z in order to write the metrics in a simplified form. Therefore constant names used in this section are unrelated to previous appearances in earlier sections. After giving explicitly the line-element (no differential equations remain to be solved) we will study

some of their properties. In particular, we will give the Petrov type, the fluid 4-velocity, the kinematical quantities, the energy-density and the pressure.

We will also study in which spacetime region the solution represents a perfect fluid. As discussed earlier, there is a symmetry between exchanging x and t in the metric coefficients and switching between a timelike perfect fluid and a tachyonic fluid. In the case of metrics A and D there are regions of both, perfect and tachyonic fluid. Here a switch $x \leftrightarrow t$ might be of interest to exchange the physically meaningful regions.

Further, we describe for which ranges of the parameters the spacetime satisfies energy conditions everywhere (in the perfect-fluid region) and whether the fluid satisfies an equation of state $p = p(\rho)$ or not.

A brief description of the curvature singularities of the spacetime and an identification of the spatially homogeneous and isotropic limits in each family are also given. All four families have the magnetic part of the Weyl tensor with respect to the fluid 4-velocity being non-zero.

Tensor components have been calculated with the help of the computer program CLASSI [15] using the orthonormal tetrad

$$\boldsymbol{\theta^\alpha} = \sqrt{\mid g_{\alpha\alpha} \mid}\boldsymbol{dx^\alpha}, \qquad\qquad \text{(no summation over } \alpha)$$

for each metric (all four line-elements are diagonal).

6.1 Analysis of the solution A

After rescaling the coordinates $\{t, x, y, z\}$ and a coordinate change $e^t \cosh(x) \to t$, $e^t \sinh(x) \to x$ the metric can be written as

$$ds^2 = S^{-2}\left(-dt^2 + dx^2 + t^{1-a-b}dy^2 + t^{1-a+b}dz^2\right)$$

where $S = t^{-a/2}x^{1-a^2/q}$ and a, b are essential constants that are free apart from the condition, that q defined as $q = b^2 + a^2 - 2a - 1$ has to be non-zero. Coordinates t, x are restricted to $0 < t$ and $0 < x$ to allow arbitrary real exponents in S. We will see below that in order to satisfy positivity conditions of energy, the parameter a has to take values for which there is a curvature singularity at $t = 0$ such that the restriction $t > 0$ is no loss of generality. As a switch $b \leftrightarrow -b$ is equivalent to $y \leftrightarrow z$, we consider only $b \geq 0$. The metric has two conformal Killing vectors

$$\vec{k_1} = \partial_x, \qquad \vec{k_2} = 2t\partial_t + 2x\partial_x + (1 + a + b)y\partial_y + (1 + a - b)z\partial_z.$$

With the further abbreviation $D = q^2x^2 - 4a^2t^2$ the fluid velocity one-form has components $u_\alpha = -\operatorname{sign}(q)D^{-1/2}(qx, 2at, 0, 0)$. As announced before, lower Greek indices denote components in the frame $\boldsymbol{\theta^\alpha}$. An overall minus sign was chosen in order to

have a future pointing $\vec{u}$. To exclude null or tachyonic fluid the coordinates have to be restricted to guarantee $D > 0$, i.e.

$$\frac{x}{t} > 2\left|\frac{a}{q}\right|. \tag{35}$$

The Petrov type of the metric is I and the non-vanishing Weyl spinor components Ψ are

$$\Psi_0 = \Psi_4 = -\frac{abS^2}{4t^2}, \quad \Psi_2 = \frac{(b^2+a-1)S^2}{12t^2}.$$

In the case $b = 0$, $a = 1$ the metric is conformally flat. Then the metric has 6 KVs, shows $\rho - p < 0$ and obeys an equation of state which is

$$\rho + p = \frac{(p-\rho)}{8}\left[\left(\frac{p-\rho}{12}\right)^2 - 1\right].$$

It therefore is a special FLRW spacetime. Otherwise, in any one of the four cases

$$a = 0, \quad b = 0, \quad b = 1, \quad b = |\, 1 - a \,|$$

the metric is of Petrov type D. The expansion of the fluid is

$$\theta = \frac{S}{2|q|txD^{3/2}}\,[48(a^2-q)a^3t^4 - 16a(a^2-q)q^2t^2x^2 + (a+2)q^4x^4].$$

The acceleration and the acceleration scalar of the fluid are

$$\begin{aligned} a_\alpha &= -\frac{(2a+q)q^2x^2S}{D^2}\,(2at, qx, 0, 0)\,, \\ a_\alpha a^\alpha &= (2a+q)^2q^4x^4S^2/D^3. \end{aligned}$$

The non-vanishing components of the shear tensor $\sigma_{\alpha\beta}$ read

$$\sigma_{00} = \frac{2at}{qx}\,\sigma_{01} = \frac{4a^2t^2}{q^2x^2}\,\sigma_{11} = -\frac{4a^2|q|txS[D(1-a) - 4at^2(2a+q)]}{3D^{5/2}},$$

$$\begin{aligned} \sigma_{22} &= \frac{|q|xS[(1-a-3b)D - 4(2a+q)at^2]}{6tD^{3/2}}, \\ \sigma_{33} &= \frac{|q|xS[(1-a+3b)D - 4(2a+q)at^2]}{6tD^{3/2}}. \end{aligned}$$

The shear scalar is

$$\sigma^{\alpha\beta}\sigma_{\alpha\beta} = \frac{q^2x^2S^2}{6t^2D^3}\left(3b^2D^2 + [4at^2(a^2+b^2-1) + (a-1)D]^2\right).$$

With the energy density

$$\rho = \frac{(a^2 - q)S^2[q^2x^2 - 12(a^2 - q)t^2]}{4q^2t^2x^2}$$

and the pressure

$$p = \frac{(a^2 - q)S^2[q^2x^2 + 4(a^2 - 3q)t^2]}{4q^2t^2x^2}$$

we obtain

$$\rho + p = \frac{(a^2 - q)S^2D}{2q^2t^2x^2}, \qquad \rho - p = \frac{2(a^2 - q)(3q - 2a^2)S^2}{q^2x^2}.$$

To have the last two quantities non-negative and therefore ρ non-negative, a and b have to satisfy

$$a \geq -1/2, \quad 2a + 1 \geq b^2 \geq 2a + 1 - a^2/3.$$

For $a > -2$ there is a big-bang like singularity at a finite distance. There are two possibilities to have an equation of state $p = p(\rho)$:

- For $q = -2a$ both, ρ and p, are functions of x/t.
- For $q = 2a^2/3$ we have $p = \rho = (a^2x^2 - 9t^2)/(12t^{a+2}x^3)$.

The metric A has an abelian three dimensional conformal subgroup and therefore belongs to a class of metrics discussed previously by A. M. Sintes in [19] and A. M. Sintes, J. Carot and A. A. Coley in [3] although no explicit expressions for the metric were given there. In [3] it is explained that the energy conditions $\rho + p > 0,\ \rho - p > 0$ can only be fulfilled in a special subclass to which metric A does not belong to. In contradiction we found that for any $a > -1/2$ there exists a range of values for b such that both energy conditions are satisfied in the complete perfect-fluid range.

6.2 Analysis of the Solution B

The metric is

$$ds^2 = \frac{1}{S^2}\left[-\frac{dt^2}{\sinh^{2+2m}(x)} + \frac{dx^2}{\sinh^{2+2m}(x)} + \frac{e^{-2t}}{\sinh^2(x)}dy^2 + \mathrm{cotanh}^2(x)dz^2\right],$$

where S depends on the variable $u \equiv t + \log\sinh(x)$. This coordinate approaches asymptotically a null coordinate for big x. The conformal Killing vector of this metric is

$$\vec{k} = \partial_t + y\partial_y.$$

We define a new function $z(u)$ as

$$\frac{dS}{du} \equiv (z-1)\,S.$$

The ordinary differential equation that $S(u)$ had to satisfy has in general two constants of integration. However, by adding appropriate constants to the coordinates t and x and rescaling all the coordinates, it can be easily seen that there only appear two inequivalent cases (we are discarding the homothetic solutions because they have been already investigated in [2]). The explicit expressions for S and z are:

$$\begin{aligned}
S &= be^{-\frac{1+m}{2}u}\cosh(\frac{\sqrt{1+m^2}}{2}u) \\
&\qquad\qquad \Rightarrow \quad z = \frac{1}{2}\left[1-m+\sqrt{1+m^2}\tanh\left(\frac{\sqrt{1+m^2}}{2}u\right)\right], \\
S &= be^{-\frac{1+m}{2}u}\sinh(\frac{\sqrt{1+m^2}}{2}u) \\
&\qquad\qquad \Rightarrow \quad z = \frac{1}{2}\left[1-m+\sqrt{1+m^2}\mathrm{cotanh}\left(\frac{\sqrt{1+m^2}}{2}u\right)\right].
\end{aligned}$$

where b is an arbitrary constant. So, the metric has two essential parameters: the constant b gives a global scale to the metric and the constant m measures the departure from the maximally symmetric anti-de Sitter spacetime (see below) and therefore is a measure of the inhomogeneity of the spacetime.

The tetrad components of the fluid velocity one-form of this solution are

$$u_\alpha = (-1, 0, 0, 0)$$

so, the metric is given in comoving coordinates. The non-vanishing Weyl spinor components Ψ are

$$\Psi_0 = \frac{mS^2}{2}\sinh^{2m+1}(x)e^x, \qquad \Psi_2 = \frac{mS^2}{6}\sinh^{2m}(x),$$
$$\Psi_4 = -\frac{mS^2}{2}\sinh^{2m+1}(x)e^{-x},$$

so that the Petrov type is I except when the constant m vanishes and the metric is conformally flat. In this case, the density and pressure are $\rho = -3b^2/4$, $p = 3b^2/4$ and therefore the spacetime is anti-de Sitter.

The expansion of the fluid is

$$\theta = S\sinh^{1+m}(x)\,(2-3z)\,.$$

The shear tensor of the solution is diagonal and its components are

$$\sigma_{11} = -\frac{1}{2}\sigma_{22} = \sigma_{33} = \frac{S}{3}\sinh^{1+m}(x), \qquad \sigma^{\alpha\beta}\sigma_{\alpha\beta} = \frac{2S^2}{3}\sinh^{2+2m}(x).$$

The acceleration of the fluid is

$$\boldsymbol{a} = -\mathrm{cotanh}(x)\,(m+z)\,\boldsymbol{dx}.$$

The density and the pressure are

$$\rho = S^2\sinh^{2m}(x)\left(m - 3z^2\right), \qquad p = S^2\sinh^{2m}(x)\,(2m+3z)\,z.$$

From these expressions it is clear that the perfect fluid does not satisfy an equation of state $p = p(\rho)$ for any value of m. The solution with S being a hyperbolic sine has always a region near $u = 0$ where the energy-density is negative. Thus, we will concentrate on the solution with S being a hyperbolic cosine. For this solution the energy-density is positive everywhere provided the constant m satisfies

$$m > \frac{4}{3}.$$

The range of variation of the coordinates for this solution is

$$-\infty < t < \infty, \qquad 0 < x, \qquad -\infty < y < \infty, \qquad -\infty < z < \infty.$$

When $u \to -\infty$ the function S tends to infinity in an exponential way. So, the length of the curve $x =$ const., $y =$ const. and $z =$ const. from any finite value of u to $u = -\infty$ is finite. Furthermore, the energy-density diverges here and therefore we have a big-bang-like singularity at $t = -\infty$. On the other hand, we have that $u = +\infty$ is located at an infinite distance and the same happens with $x = 0$, which is also located at infinity. The expansion of the fluid is positive everywhere and tends to infinity when we approach the big bang and to zero in the infinite future.

Let us now study in which region the pressure is positive. A trivial calculation shows that the pressure is positive in either of the following regions

$$\tanh\left(\frac{\sqrt{1+m^2}}{2}u\right) \geq \frac{m-1}{\sqrt{1+m^2}} \quad \text{or} \quad \tanh\left(\frac{\sqrt{1+m^2}}{2}u\right) \leq -\frac{3+m}{3\sqrt{1+m^2}}.$$

Furthermore, in the range for m we are interested in $(m > 4/3)$, we have

$$\frac{m-1}{\sqrt{1+m^2}} < 1 \qquad \text{and} \qquad -\frac{3+m}{3\sqrt{1+m^2}} > -1,$$

and, therefore, there always exist two values of u, say u_0 and u_1, such that the pressure is positive for

$$-\infty < u \leq u_0 < 0 \qquad \text{and} \qquad 0 < u_1 \leq u < +\infty,$$

i.e. it is positive except for a finite interval around $u = 0$.

6.3 Analysis of the solution C

After rescaling $\{t, x, y, z\}$ and a coordinate change $e^{-x} \to x$ we find that the metric takes the form

$$ds^2 = \frac{1}{S^2(x,t)} \left[-dt^2 + \frac{dx^2}{x^2} + x^{b(b+1)} \cosh^{1-b}(t) dy^2 + x^{b(b-1)} \cosh^{1+b}(t) dz^2 \right], \qquad (36)$$

where the function $S(x,t)$ is

$$S(x,t) = x^{\frac{1+b^2}{2}} + s_0 \mid \sinh t \mid^{\frac{1+b^2}{2}}, \qquad s_0 \ \text{constant}.$$

As a switch $b \leftrightarrow -b$ is equivalent to $y \leftrightarrow z$, we consider only $b \geq 0$. It is convenient to define the two functions $S_x(x)$ and $S_t(t)$ as

$$S_x \equiv x^{\frac{1+b^2}{2}}, \qquad S_t \equiv s_0 \mid \sinh t \mid^{\frac{1+b^2}{2}}, \qquad (S = S_x + S_t).$$

The conformal Killing vector of this solution is

$$\vec{k} = x\partial_x - \frac{1}{2} b\,(1+b)\, y\partial_y + \frac{1}{2} b\,(1-b)\, z\partial_z.$$

The non-vanishing Weyl spinor components Ψ are

$$\Psi_0 = \frac{S^2}{4\cosh t} b\left(1-b^2\right) e^{-t}, \quad \Psi_2 = \frac{S^2}{12\cosh^2 t}\left(1-b^2\right), \quad \Psi_4 = \frac{S^2}{4\cosh t} b\left(1-b^2\right) e^{t}.$$

Consequently, the Petrov type of this metric (when $b \neq 1$ and $b \neq 0$) is I everywhere except at the spacelike hypersurface $\cosh t = \mid b^{-1} \mid$ where it degenerates to type D. In the particular case $b = 1$, the metric is conformally flat and the density and pressure of the perfect fluid are

$$\rho = 3s_0^2, \qquad p = -3s_0^2,$$

so that the metric is the de Sitter spacetime. When $b = 0$ the Petrov type is D everywhere.

Let us now analyze the kinematical quantities and the matter content for the metric (36). The tetrad components of the fluid velocity one-form of this solution is

$$u_\alpha = (-1, 0, 0, 0)$$

and the metric is written in comoving coordinates.

The fluid expansion is

$$\theta = \frac{1}{2\cosh t \sinh t} \left[2S \sinh^2 t - 3S_t \left(1+b^2\right) \cosh^2 t \right].$$

The non-vanishing components of the shear tensor and the shear scalar are

$$\sigma_{11} = -\frac{S}{3}\tanh t, \qquad \sigma_{22} = \frac{S}{6}\,(1-3b)\tanh t,$$

$$\sigma_{33} = \frac{S}{6}\,(1+3b)\tanh t, \qquad \sigma_{\alpha\beta}\sigma^{\alpha\beta} = \frac{S^2}{6}\tanh^2 t\left(1+3b^2\right).$$

The only non-zero tetrad component of the acceleration is

$$a_1 = -\frac{b^2+1}{2}S_x.$$

The expressions for the energy-density and pressure read

$$\rho = \frac{(b^2-1)\,S^2\sinh^2 t + 3\,(b^2+1)^2\,S_t^2\cosh^2 t}{4\sinh^2 t\cosh^2 t},$$

$$p = \frac{(b^2-1)\,S^2\sinh^2 t + (b^2+1)\,S_t\cosh^2 t\left[\,2\,(b^2-1)\,S_x - (b^2+5)\,S_t\right]}{4\sinh^2 t\cosh^2 t}.$$

The perfect fluid does not satisfy an equation of state $p = p(\rho)$ unless $b = 1$ (which corresponds to the de Sitter spacetime, as discussed above) or when $s_0 = 0$. In the latter case the perfect fluid is a stiff fluid ($p = \rho$) and the metric coefficients are separable functions in comoving coordinates. Therefore, this particular case belongs to the class of metrics studied in [1] where all the G_2 on S_2 diagonal stiff fluid metrics with separable metric coefficients were considered.

In order to find the ranges of variation of the coordinates $\{t, x, y, z\}$ which define the spacetime, we must distinguish a number of different cases depending on the signs of s_0 and of (b^2-1). The reason is that if $b^2-1>0$ then the spacetime has no curvature singularity at $t=0$ and it can be continued across $t=0$ to negative values of t. On the other hand, when $0\leq b^2<1$ then the energy-density blows up at $t=0$ and we have a big bang singularity there. Thus, the two possible variation ranges for the coordinate t are

$$\begin{array}{rl} -\infty < t < +\infty, & \text{if } \; b^2 > 1 \\ 0 < t < +\infty & \text{if } \; 0 \leq b^2 < 1. \end{array}$$

Regarding the parameter s_0, we must impose that the function S be non-zero (it can be either positive or negative as only S^2 appears in the metric). When $s_0 \geq 0$, x can take arbitrarily large values, but when $s_0 < 0$ we have that two different variation ranges are possible (they correspond respectively to the region where $S > 0$ and to the region where $S < 0$). Thus, the different possibilities for the variation range for the coordinate x are

$$\begin{array}{rl} 0 < x < \infty, & \text{if } \; s_0 \geq 0 \\ 0 < x < (-s_0)^{\frac{2}{1+b^2}}\mid\sinh t\mid & \text{if } \; s_0 < 0 \\ (-s_0)^{\frac{2}{1+b^2}}\mid\sinh t\mid < x < +\infty & \text{if } \; s_0 < 0. \end{array}$$

These different variation ranges define completely different spacetimes, they are not different regions of a single spacetime. Regarding the coordinates y and z, their variation range is, in all cases,

$$-\infty < y < +\infty, \qquad -\infty < z < +\infty.$$

We can now specify which of these different possibilities satisfy $\rho > 0$, $\rho - p > 0$ and $\rho + p > 0$ in the whole spacetime. A simple analysis shows that this can be accomplished only in one case, namely, when

$$b > 1, \qquad s_0 < 0,$$

$$0 < t < +\infty, \qquad 0 < x < (-s_0)^{\frac{2}{1+b^2}} \mid \sinh t \mid,$$

$$-\infty < y < +\infty, \qquad -\infty < z < +\infty$$

(we have to restrict t to $t > 0$ because of the variation range of x). There is no curvature singularity anywhere in the spacetime and so we come to the question of whether it can be extended to a larger spacetime or not. The vanishing of S at the boundary

$$x = (-s_0)^{\frac{2}{1+b^2}} \mid \sinh t \mid$$

suggests that the boundary is located at an infinite distance and therefore the spacetime can not be extended across this boundary (a formal proof of completeness is more complicated, however). Regarding the points with $x = 0$ the Riemann tensor is regular there. The distance to these points along a curve $t =$ const., $y =$ const. and $z =$ const. is divergent, which again suggests that they are located at infinity. This does not prove, however, that the spacetime is complete although it is an indication for it. Thus, this spacetime has no singularities and satisfies the dominant energy conditions [12] everywhere. Unfortunately, the strong energy condition $\rho + 3p \geq 0$ is violated in some regions of the spacetime.

6.4 Analysis of the solution D

The metric is

$$ds^2 = \frac{e^{-(1+a)r}}{S^2(r,t)} \left[-dt^2 + q(r)dr^2 + e^{-2t}dy^2 + \frac{e^{2ar}}{q(r)}dz^2 \right], \tag{37}$$

where a is an arbitrary parameter and $q(r)$ is given by

$$q(r) \equiv \frac{aAe^{2ar}}{1 + Ae^{2ar}}, \qquad A \text{ non-vanishing constant} \tag{38}$$

(From the positivity of q follows that A must be negative when $a < 0$). The subcase $a = 0$ is included by setting $A = -1$ and performing the limit $a \to 0$ which gives $q = 1/(2r)$. The range of variation of the coordinates y and z are

$$-\infty < y < \infty, \qquad -\infty < z < \infty,$$

while the range of variation for the coordinate r depends on the sign of A

$$A > 0 \implies -\infty < r < +\infty, \qquad A < 0 \implies r > 0.$$

The function S in the metric depends on the variable $v \equiv t + r$ and satisfies the differential equation

$$\frac{(a+1)}{4}\frac{S_{,v}}{S} + \frac{(3+a)}{4}\frac{{S_{,v}}^2}{S^2} + \frac{S_{,v}S_{,vv}}{S^2} + \frac{(a+1)}{4}\frac{S_{,vv}}{S} = 0. \tag{39}$$

It can be simplified by defining a new function $w(v)$ as

$$\frac{dS}{dv} \equiv \frac{(a+1)\,S}{2\,(w-1)} \qquad a \neq -1, \tag{40}$$

(the case $a = -1$ has to be considered separately but we will not study it here in any detail as it has negative pressure everywhere). With this equation (39) becomes

$$\frac{dw}{dv} = \frac{w\,(a+w)}{w+1}, \tag{41}$$

which is trivial to integrate (we will not need the explicit solution, though). Furthermore, S can be integrated explicitly in terms of w from (40) as

$$S = b\frac{|\, w - 1 \,||\, w + a \,|^{\frac{1-a}{2a}}}{|\, w \,|^{\frac{1+a}{2a}}},$$

where b is an arbitrary positive constant which gives a global scale to the spacetime. This expression for S is also valid when $a = 0$ by performing the limit $a \to 0$. The result is

$$S = b\frac{|\, w - 1 \,|}{|\, w \,|}e^{\frac{1}{2w}}.$$

The conformal Killing vector of the metric (37) reads

$$\vec{k} = \partial_t + y\partial_y.$$

The Petrov type is D everywhere with the non-vanishing Weyl spinor components Ψ given by

$$\Psi_0 = -3\Psi_2 = \Psi_4 = \frac{(a-1)}{4}S^2 e^{(a+1)r}.$$

In the particular case $a = 1$ the metric is conformally flat and the perfect fluid has $\rho = -3b^2/A$ and $p = 3b^2/A$. Thus, the spacetime in this limiting case is de Sitter ($A < 0$), anti-de Sitter ($A > 0$) or Minkowski ($A = \infty$, which gives $q = 1$).

Returning to the general case $a \neq 1$, the two repeated null principal directions of the Weyl tensor are

$$\boldsymbol{l} = \frac{1}{\sqrt{2}} \frac{e^{-\frac{(a+1)r}{2}}}{S} \left(\boldsymbol{dt} + e^{-t}\boldsymbol{dy}\right), \qquad \boldsymbol{k} = \frac{1}{\sqrt{2}} \frac{e^{-\frac{(a+1)r}{2}}}{S} \left(\boldsymbol{dt} - e^{-t}\boldsymbol{dy}\right).$$

The tetrad components of the fluid velocity one-form are

$$u_\alpha = \frac{1}{\sqrt{q - w^2}}(-\sqrt{q}, w, 0, 0)$$

and therefore the fluid velocity vector does not lie in the two-plane generated by the two repeated null principal directions at each point of the spacetime.

The condition for the existence of a timelike $\vec{u}$ is $q > w^2$ (in the region of the spacetime where $q = w^2$ the energy-momentum tensor represents a null fluid and in the region $0 < q < w^2$ the matter content is a tachyon fluid, $q < 0$ is forbidden in order to preserve signature).

The expansion of the fluid is

$$\theta = \sqrt{\frac{q - w^2}{q}} \frac{S e^{\frac{(1+a)r}{2}}}{2(q - w^2)^2(w - 1)} \Big[-3a(q + w^2)(q - w^2) + q^2(4w^2 - 6w - 1) - \\ - 2qw^2(w - 1)(w - 2) + 3w^4 \Big],$$

the non-vanishing components of the shear tensor and the shear scalar are

$$\sigma_{00} = \frac{w^2}{q}\sigma_{11}, \quad \sigma_{01} = -\frac{w}{\sqrt{q}}\sigma_{11}, \quad \sigma_{33} + \sigma_{22} + \frac{q - w^2}{q}\sigma_{11} = 0,$$

$$\sigma_{11} = \frac{S}{3} e^{\frac{(1+a)r}{2}} q^2 \sqrt{\frac{q - w^2}{q}} \frac{(q + w^2)(w + 1)}{(q - w^2)^3},$$

$$\sigma_{22} = \frac{S}{3} e^{\frac{(1+a)r}{2}} q \sqrt{\frac{q - w^2}{q}} \frac{(w^2 - 2q)(w + 1)}{(q - w^2)^2},$$

$$\sigma_{\alpha\beta}\sigma^{\alpha\beta} = \frac{2}{3} e^{(a+1)r} S^2 q (w + 1)^2 \frac{(q - w^2)^2 + qw^2}{(q - w^2)^3}.$$

The non-zero tetrad components of the acceleration and the acceleration scalar of the fluid are

$$a_0 = -\frac{w}{\sqrt{q}} a_1 = \frac{e^{\frac{(1+a)r}{2}} S w^2}{(q - w^2)^2(w^2 - 1)} \Big[2a(q - w^2) - q(w^3 + w^2 - 3w - 1) - 2w^3 \Big],$$

$$a_\alpha a^\alpha = a_1^2 \frac{(q-w^2)}{q}.$$

The energy-density and the pressure read

$$\begin{aligned}
\rho &= \frac{e^{(1+a)r}S^2}{4(w-1)^2 q}\left[4q(w-1)\,(aw+2w+2a+1)+3(a+1)^2(q-w^2)\right],\\
p &= \frac{e^{(1+a)r}S^2}{(w-1)^2 q}\left[q(1-w)\,(aw+2w+2a+1)-\frac{(a+1)(q-w^2)\,(5aw+w+a+5)}{4(w+1)}\right].
\end{aligned}$$

It can be easily seen that the perfect fluid does not satisfy an equation of state $p = p(\rho)$ unless $a = 1$ (i.e. the de Sitter, anti-de Sitter and Minkowski limits of the family, see above). The expression for $\rho + p$ is simple and reads

$$\rho + p = \frac{S^2}{2} e^{(1+a)r} \frac{(q-w^2)(a^2-1)}{(1-w^2)q}$$

and, therefore, in the region where the fluid velocity is timelike we have that

$$\rho + p > 0 \quad \Longleftrightarrow \quad \frac{a^2-1}{1-w^2} > 0.$$

In order to study the range of variation of the function w it is convenient to consider the metric (37) using the coordinates $\{w, r, y, z\}$. We will only need the coefficient in dw^2 which reads

$$-e^{-(1+a)r}\frac{(w+1)^2}{b^2(w-1)^2}\frac{\mid w \mid^{\frac{1-a}{a}}}{\mid w+a \mid^{\frac{a+1}{a}}}dw^2.$$

Analyzing this coefficient together with the expression for $\rho+p$ we find that the function w is allowed to vary in five different ranges. The extrema of the intervals are

$$w \to -\infty, \quad w = -1, \quad w = 0, \quad w = 1, \quad w = a, \quad w \to +\infty$$

where a may take an arbitrary value. Each one of the five possibilities gives a different behaviour for the metric and the matter content. Let us restrict the possibilities by imposing $\rho + p > 0$ and $\rho > 0$ everywhere in the region where the matter content is a perfect fluid (that is, in the region $q > w^2$). It turns out that there exist four different cases in which these two conditions are fulfilled. The first one is

$$a < -1, \qquad -1 < w < 0.$$

In this case both hypersurfaces $w = -1$ and $w = 0$ are curvature singularities of the spacetime and they are located at a finite distance. The second possibility is

$$a < -1, \qquad 0 < w < 1.$$

Here, $w = 0$ is a singularity of the spacetime located at a finite distance and $w = 1$ is at an infinite distance in the future. The third one is

$$-1 < a < 1, \qquad w < -1,$$

now the spacetime has a future singularity at $w = -1$ at a finite distance and $w = -\infty$ is at the infinite past. Finally,

$$-1 < a < 1, \qquad w > 1,$$

which has a curvature singularity at $w = 1$ at a finite time in the past and extends infinitely into the future.

Acknowledgements

The authors wish to thank A. Koutras for the help with the computer algebra program CLASSI. M.M wishes to thank J. Senovilla for helpful suggestions related with this work and the Ministerio de Educación y Ciencia for financial support.

7 Appendix

The aim of this appendix is the analysis of the equation

$$\Sigma_1(x) + \frac{\dot{H}}{H}\Sigma_2(x) + \frac{\dot{H}^2}{H^2}\Sigma_3(x) + \frac{\ddot{H}}{H}\frac{\dot{H}}{H}\Sigma_4(x) + \frac{\ddot{H}}{H}\Sigma_5(x) = 0, \tag{42}$$

which involves two independent variables, x and $t + bK(x)$. Let us first assume that $\Sigma_4 \equiv 0$. Then the equation reads

$$\Sigma_1(x) + \frac{\dot{H}}{H}\Sigma_2(x) + \frac{\dot{H}^2}{H^2}\Sigma_3(x) + \frac{\ddot{H}}{H}\Sigma_5(x) = 0.$$

If, in addition, $\Sigma_5 \equiv 0$ then this equation is a polynomial of second degree in $\dot{H}/H$ which can be solved (unless the polynomial is identically zero) giving $\dot{H}/H$ (which is a function only of $t+bK(x)$) in terms of functions of only x. This implies that $\dot{H}/H$ must be constant, thus producing homothetic solutions, against our assumptions. Thus, the only possibility is

$$\Sigma_1 = \Sigma_2 = \Sigma_3 = \Sigma_4 = \Sigma_5 = 0, \qquad H \quad \text{arbitrary}$$

which we label as Case 1.

When $\Sigma_4 \equiv 0$ and $\Sigma_5 \neq 0$, we can divide equation (42) by Σ_5 and apply the differential operator $\vec{D} \equiv \partial_x - bK'\partial_t$ which kills all functions depending only on the variable $t + bK(x)$. The result is

$$\left(\frac{\Sigma_1}{\Sigma_5}\right)' + \left(\frac{\Sigma_2}{\Sigma_5}\right)' \frac{\dot{H}}{H} + \left(\frac{\Sigma_3}{\Sigma_5}\right)' \frac{\dot{H}^2}{H^2} = 0.$$

Now, the same considerations made before imply that each coefficient in this polynomial in $\dot{H}/H$ must be zero. Thus, we necessarily have

$$\Sigma_4 = 0, \quad \Sigma_1 = m_1\Sigma_5, \quad \Sigma_2 = m_2\Sigma_5, \quad \Sigma_3 = m_3\Sigma_5, \quad \Sigma_5 \neq 0,$$
$$m_1 + m_2\frac{\dot{H}}{H} + m_3\frac{\dot{H}^2}{H^2} + \frac{\ddot{H}}{H} = 0,$$

where m_1, m_2 and m_3 are arbitrary constants. We label this possibility as Case 2. We have finished the analysis when $\Sigma_4 \equiv 0$, so let us assume from now on that $\Sigma_4 \neq 0$. Dividing (42) by Σ_4 and applying the operator $\vec{D}$ on the resulting equation, we immediately find

$$\left(\frac{\Sigma_1}{\Sigma_4}\right)' + \left(\frac{\Sigma_2}{\Sigma_4}\right)' \frac{\dot{H}}{H} + \left(\frac{\Sigma_3}{\Sigma_4}\right)' \frac{\dot{H}^2}{H^2} + \left(\frac{\Sigma_5}{\Sigma_4}\right)' \frac{\ddot{H}}{H} = 0. \tag{43}$$

When $(\Sigma_5/\Sigma_4)' \equiv 0$ we necessarily have

$$\Sigma_1 = k_1\Sigma_4, \quad \Sigma_2 = k_2\Sigma_4, \quad \Sigma_3 = k_3\Sigma_4, \quad \Sigma_5 = k_5\Sigma_4, \quad \Sigma_4 \neq 0$$
$$k_1 + k_2\frac{\dot{H}}{H} + k_3\frac{\dot{H}^2}{H^2} + \frac{\dot{H}}{H}\frac{\ddot{H}}{H} + k_5\frac{\ddot{H}}{H} = 0,$$

(k_1, k_2, k_3 and k_5 arbitrary constants), which is the possibility labeled as Case 3. It only remains the situation when $(\Sigma_5/\Sigma_4)' \neq 0$. We can divide equation (43) by $(\Sigma_5/\Sigma_4)'$ and apply the differential operator $\vec{D}$ on the resulting equation. We find a polynomial involving only $\dot{H}/H$ which, as usual, implies that each coefficient must vanish. The result is

$$\Sigma_1 = k_1\Sigma_5 + \beta_1\Sigma_4, \qquad \Sigma_2 = \alpha_2\Sigma_5 + \beta_2\Sigma_4, \qquad \Sigma_3 = \alpha_3\Sigma_5 + k_3\Sigma_4,$$

where k_1, k_3, α_2, α_3, β_1 and β_2 are constants. Equation (43) is

$$k_1 + \alpha_2\frac{\dot{H}}{H} + \alpha_3\frac{\dot{H}}{H} + \frac{\ddot{H}}{H} = 0.$$

It still remains to impose the original equation (42) which, using all this information, reads

$$\beta_1 + (\beta_2 - k_1)\frac{\dot{H}}{H} + (k_3 - \alpha_2)\frac{\dot{H}^2}{H} - \alpha_3\frac{\dot{H}^3}{H^3} = 0.$$

Each coefficient in this polynomial must vanish and we finally find

$$\Sigma_1 = k_1\Sigma_5, \quad \Sigma_2 = k_1\Sigma_4 + k_3\Sigma_5, \quad \Sigma_3 = k_3\Sigma_4, \quad \Sigma_5 \neq 0, \quad \frac{\Sigma_4}{\Sigma_5} \neq \text{const.}$$
$$k_1 + k_3\frac{\dot{H}}{H} + \frac{\ddot{H}}{H} = 0,$$

which is the last possibility, labeled as Case 4.

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